Inner Amenable Groups

Amenability

Def A group Γ is amenable if it admits a finitely additive prob. meas. (mean) $m: P(\Gamma) \to [0,1]$ that is invariant under left translation, i.e. $m(\gamma A) = m(A)$

Def An action $\Gamma \curvearrowright X$ is amenable if $\exists$ a mean on X that is Γ-invariant.

Eg $\Gamma \curvearrowright \Gamma$ by left translation is amenable iff Γ amenable.

Prop If Γ is amenable, then $\Gamma \curvearrowright X$ is amenable.

Eg. If Γ is nonamenable, $\Lambda \trianglelefteq \Gamma$ s.t. Γ/Λ is amenable.

then $\underbrace{\Gamma \curvearrowright \Gamma/\Lambda \text{ by left translation is amenable.}}_{\text{coamenable}}$

$\mathbb{F}_2 \leq \Gamma$, does there exist $\Lambda \cong \mathbb{F}_2$
$\Lambda \leq \Gamma$ non-coamenable

Prop If Γ is nonamenable, and

$\Gamma \curvearrowright X$ amenable with mean m on X

then Γ_x is nonamenable for m-almost every $x \in X$

Pf: $\Gamma \curvearrowright X$ amenable

Γ_x amenable $\forall x$

Let X_0 be a transversal. For $x \in X_0$, let ν_x be a mean for $\Gamma_x \curvearrowright \Gamma$

Extend to X by $\nu_{g \cdot x} = g \cdot \nu_x$

Define $\nu(A) = \int_X \nu_x(A)\, dm(x)$ $\square$

Def A group Γ is inner amenable if it admits a diffuse conjugation-invariant mean.

i.e. $m(D) = 0$ if D is finite.

E.g.
- infinite amenable groups
- $\Lambda \times \Delta$ where Δ is inner amenable

  $\{e\} \times \Delta$ is conj-inv.

  Λ
- Γ has inf. center (or inf. FC-center)
- Γ has asymp. central sequence (diffuse)

  $(F_i) \subseteq \Gamma$ finite sets s.t. $\forall \gamma \in \Gamma$ $\gamma\delta = \delta\gamma$ for all $\delta \in F_i$ if i is large enough
- $\Gamma = \bigoplus_{i \in \mathbb{N}} \Gamma_i$
- certain top. full group (Kerr-Tucker-Drob)

nonexamples
- $\mathbb{F}_2$
- properly proximal gps.